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DUALITY FOR RATIONAL NORMAL SCROLLS

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1. Introduction

Given a skew curve $C \subset \mathbb{P}^n$, its strict dual curve $C^* \subset \check{\mathbb{P}}^n$ is defined as the closure of the set of hyperplanes that are osculating to C of order n at smooth points of C. The strict dual of C^* is C, and there is a natural duality between the various osculating developables of C and those of C^* [P1, §5]. If $C \subset \mathbb{P}^n$ is a rational normal curve, then so is $C^* \subset \check{\mathbb{P}}^n$; we say that C is self-dual.

In this paper we shall consider rational normal scrolls $X \subset \mathbb{P}^N$ of dimension r, and study their higher order dual varieties $X_m^\vee \subset \check{\mathbb{P}}^N$, in particular the strict dual variety $X^* \subset \check{\mathbb{P}}^N$. We show that only the generic scrolls (in the sense of Ghione [G]) satisfy the biduality $X^{**} = X$, and that, among these, only the balanced scrolls are self-dual. By defining the "restriction" of X^* to sub-scrolls of X, the nature of the self-duality for balanced scrolls appears even clearer.

Similar considerations could be applied to non-rational scrolls and to non-normal scrolls (e.g., projections of normal ones). Except for statements like: "if there exists a directrix of low degree, then the dual varieties are contained in a lower-dimensional space, and so biduality does not hold", it is not so simple, however, to state general results in those cases.

2. Rational normal scrolls

Fix integers $0 \leq d_1 \leq \cdots \leq d_r$, with $d_r > 0$, and consider the rank r bundle

$$F = \bigoplus_{i=1}^{r} O_{\mathbb{P}^1}(d_i)$$

on $\mathbb{P}^1 = \mathbb{P}^1_k$, where k is an algebraically closed field of characteristic 0. The complete linear system associated with the line bundle $O_{\mathbb{P}(F)}(1)$ defines a morphism

$$\phi : \mathbb{P}(F) \to \mathbb{P}(V) \simeq \mathbb{P}^N$$

where $V = \bigoplus_{i=1}^{r} V_i$, $V_i = H^0(O_{\mathbb{P}^1}(d_i))$, and $N = \sum_{i=1}^{r}(d_i+1)-1 = \sum_{i=1}^{r} d_i + r-1$. Set $X = \phi(\mathbb{P}(F))$; then we have $\deg X = \sum_{i=1}^{r} d_i$. We shall say that X is a <u>variety of type</u> $(d_1,\ldots,d_r)$.

If $d_1 > 0$ holds, $O_{\mathbb{P}(F)}(1)$ is very ample and ϕ is an embedding. Then we call X a <u>rational normal scroll</u> of dimension r. If $d_1=\cdots=d_s= 0 < d_{s+1}$ for some s, $1 \leq s < r$, then X is a cone, with a $\mathbb{P}^{s-1}$ as vertex, over a rational normal scroll of type $(d_{s+1},\ldots,d_r)$.

Classically (C. Segre [S1],[S2]) these varieties were constructed in the following way: Take linear subspaces $\mathbb{P}^{d_i} \subset \mathbb{P}^N$, $i = 1,\ldots,r$, such that no $\mathbb{P}^{d_i}$ intersects the space spanned by the other $\mathbb{P}^{d_j}$'s. Consider now the d_i-uple embedding $\phi_i : \mathbb{P}^1 \to \mathbb{P}^{d_i}$; thus we have in each $\mathbb{P}^{d_i}$ a rational normal curve $D_i = \phi_i(\mathbb{P}^1)$. (If $d_i = 0$, D_i is just the point $\mathbb{P}^0$.) The scroll X is then the r-dimensional variety swept out by the $(r-1)$-dimensional subspaces $X_t = [\phi_1(t),\ldots,\phi_r(t)]$, as t varies in $\mathbb{P}^1$. We call the X_t's the <u>generators</u> of X.

Finally, we shall also use the notation

$$X = (D_1,\ldots,D_r) .$$

3. Higher order dual varieties

Recall the definition [P2] of higher order dual varieties of a given variety $X \subset \mathbb{P}(V) \simeq \mathbb{P}^n$: For each $m \geq 0$, consider the homomorphism

$$a^m: V_X \to P_X^m(1) ,$$

where $P_X^m(1)$ is the sheaf of principal parts of order m of $O_X(1)$. Let $s = s(m)$ denote the generic rank of a^m. If $s \leq n$, then almost all points of X has a well-defined $(s-1)$-dimensional osculating space of order m. The m-th dual variety $X_m^\vee \subset \mathbb{P}(V^\vee) \simeq \check{\mathbb{P}}^n$ of X is then defined as the closure of the set of hyperplanes containing an m-th order osculating $(s-1)$-space.

Set $\bar{s} = \max\{s(m) | s(m) \leq n, m \geq 0\}$, and $\bar{m} = \min\{m \mid s(m) = \bar{s}\}$. Then we call $X^* = X_{\bar{m}}^\vee$ the <u>strict dual</u> variety of X. This definition, being slightly more general than the one given in [P2], permits us to define the strict dual of any variety $X \subset \mathbb{P}^n$.

Assume now that $X \subset \mathbb{P}^n$ is a variety of type $(d_1,\ldots,d_r)$. The coordinates $X_0^i,\ldots,X_{d_i}^i$ on $\mathbb{P}^{d_i}$ give coordinates $X_0^1,\ldots,X_{d_1}^1,X_0^2,\ldots,X_{d_r}^r$ on $\mathbb{P}^N$. If t is an affine parameter on $\mathbb{P}^1$, then X is given parametrically at points $x \in X$ around the generator X_t by

$$x_j^i = x_j^i(\lambda_1,\ldots,\lambda_r;t) = \lambda_i t^j$$

$(i = 1,\ldots,r;\ j = 0,1,\ldots,d_i)$, where $(\lambda_1,\ldots,\lambda_r) \in \mathbb{P}^{r-1}$.

Since X has dimension r, the sheaf $P_X^m(1)$ has generic rank $\binom{t+m}{m}$. (If X is smooth, i.e., if $d_1 > 0$, then $P_X^m(1)$ is a bundle with this rank.) Since X is ruled (the parametrization is linear in the λ_i's), the generic rank of

$$a^m: V_X \to P_X^m(1)$$

is, however, always $\leq rm+1$, and clearly equality holds if and only if $m \leq d_1$. Assume, say, that $\lambda_r \neq 0$. Then the <u>image</u> of a^m is given by the matrix

$$A_m = \begin{pmatrix} x_{\lambda_1}^{(h_1)} \\ \cdot \\ \cdot \\ \cdot \\ x_{\lambda_{r-1}}^{(h_{r-1})} \\ x^{(h_r)} \end{pmatrix}$$

where $h_i = 0,1,\ldots,\min\{d_i,m-1\}$ for $1 \leqslant i \leqslant r-1$ and $h_r = 0,1,\ldots,\min\{d_r,m\}$, and where we set

$$x = (x_j^i),\ x^{(h)} = \frac{\partial^h x}{\partial t^h}\ ,\ x_{\lambda_i}^{(h)} = \frac{\partial x^{(h)}}{\partial \lambda_i}\ .$$

We shall now rewrite this matrix A_m which determines $\check{X}_m$, using the following notation:

Let $D \subset \mathbb{P}^d$ be the rational normal curve $x_i = t^i$, $i = 0,1,\ldots,d$. For each $m \geqslant 0$, set

$$M_d^m = \begin{pmatrix} x \\ x' \\ \cdot \\ \cdot \\ \cdot \\ x^{(m)} \end{pmatrix} .$$

If $m \leqslant d$, then

$$M_d^m = \begin{pmatrix} 1 & t & \cdots & & & & t^d \\ 0 & 1 & & & & & \cdot \\ \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & \cdot \\ 0 & \cdots & & 0 & 1 & \cdots & \binom{d}{m} t^{d-m} \end{pmatrix}$$

and if $m > d$, then

$$M_d^m = \begin{pmatrix} M_d^d \\ \\ 0 \end{pmatrix}$$

We shall write

$$\bar{M}_d^m = \begin{cases} M_d^m & \text{if } m \leq d \\ M_d^d & \text{if } m > d \end{cases}$$

The matrix M_d^m corresponds to the map

$$a^m : V_D \to P_D^m(1)$$

giving the m-th order osculating spaces (or the m-th osculating developable, or the m-th dual variety) of $D \subset \mathbb{P}^d$. Note that a^d is an isomorphism.

Returning to $X \subset \mathbb{P}^N$, the matrix A_m can now be written as

$$A_m = \begin{pmatrix} \bar{M}_{d_1}^{m-1} & & & & 0 & \\ & \bar{M}_{d_2}^{m-1} & & & & \\ & & \cdot & & & \\ & 0 & & \cdot & & \\ & & & & \bar{M}_{d_{r-1}}^{m-1} & \\ \lambda_1 M_{d_1}^m & \cdot & \cdot & \cdot & \lambda_{r-1} M_{d_{r-1}}^m & M_{d_r}^m \end{pmatrix}$$

for $1 \leq m \leq \bar{m}$. It follows that the m-th order osculating space to X at a point $x \in D_r \cap X_t$, is equal to the space spanned by the $(m-1)$-th order osculating space to D_i at $x_i = D_i \cap X_t$, $i = 1,\ldots,r-1$, and the m-th order osculating space to D_r at x. (If $m-1 > d_i$, then any $(m-1)$-th order osculating space to D_i is the whole space $[D_i]$.)

Similar considerations apply to the cases $\lambda_i \neq 0$. In particular, we observe that the intersection of the m-th order osculating spaces to X at points of a generator X_t contains the space spanned by the $(m-1)$-th order osculating spaces to the directrices D_i at the points $x_i = D_i \cap X_t$.

Before stating the general results, we shall give some, hopefully illuminating, examples.

4. Examples

We shall first consider some surfaces.

Example 1. $r = 2$, $\deg X = 4$, $N = 5$.

There are three possible types:

a) $d_1 = 0$, $d_2 = 4$

b) $d_1 = 1$, $d_2 = 3$

c) $d_1 = d_2 = 2$.

Case a): For $1 \leq m \leq \bar{m} = 3$, A_m becomes

$$\begin{pmatrix} M_0^{m-1} & 0 \\ \lambda_1 M_0^m & M_4^m \end{pmatrix}.$$

The strict dual variety $X^* = X_3^\vee$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & M_4^3 \end{pmatrix}.$$

The points of X^* are thus the 4-spaces spanned by an osculating 3-space to D_2 and the point D_1. In other words,

$$X = (D_2)_3^\vee \cap [D_1]^\vee.$$

Note that $(D_2)_3^\vee$ is a cone with vertex $[D_2]^\vee$. So X^* is a rational normal curve of degree 4 in the hyperplane $[D_1]^\vee \subset \check{\mathbb{P}}^5$.

Moreover, we observe that

$$X_2^\vee = (D_2)_2^\vee \cap [D_1]^\vee$$

is the tangent developable of $X^* \subset [D_1]^\vee$, and that

$$X^\vee = X_1^\vee = (D_2)_1^\vee \cap [D_1]^\vee$$

is the osculating developable of $X^* \subset [D_1]^\vee$ ([P1], 5.2).

Case b): For $1 \leq m \leq \bar{m} = 2$, A_m becomes

$$\begin{pmatrix} M_1^{m-1} & 0 \\ \lambda_1 M_1^m & M_3^m \end{pmatrix},$$

so $X^* = X_2^\vee$ is given by the matrix (equivalent to A_m)

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & \\ \hline & 0 & M_3^2 \end{array}\right)$$

Hence $X^* = (D_2)_2^\vee \cap [D_1]^\vee$ is a rational normal curve of degree 3.

Case c): For $1 < m < \bar{m} = 2$, A_m is

$$\begin{pmatrix} M_2^{m-1} & 0 \\ \lambda_1 M_2^m & M_2^m \end{pmatrix}$$

At this point it is convenient to introduce the "restriction of $X^* = X_2^\vee$ to D_i":

$X^*|D_i$ = the points of X^* corresponding to points on D_i.

Then $X^*|D_2$ is obtained by setting $\lambda_1 = 0$, hence

$$X^*|D_2 = (D_1)_1^\vee \cap [D_2]^\vee.$$

Similarly,

$$X^*|D_1 = (D_2)_1^\vee \cap [D_1]^\vee.$$

We shall call $D_i^* = (D_i)_{d_i-1}^\vee \cap [D_{3-i}]^\vee$, $i = 1,2$, the <u>strict dual curve</u> of D_i (with respect to D_{3-i}). (Note that there is a self-duality for rational normal curves between $D_i \subset [D_i]$ and $D_i^* \subset [D_i^*] = [D_{3-i}]^\vee$.)

With this notation we have

$$X^*|D_i = D_{3-i}^*, \quad i = 1,2,$$

and

$$X^* = (D_1^*, D_2^*).$$

Hence X^* is again of type $(2,2)$, and so X is self-dual.

<u>Example 2.</u> $r = 2$, $\deg X = 5$, $N = 6$.

There are three possible types:

a) $d_1 = 0$, $d_2 = 5$

b) $d_1 = 1$, $d_2 = 4$

c) $d_1 = 2$, $d_2 = 3$

The cases a) and b) are similar to the case a) and b) of Ex.1. Here we have $\bar{m} = d_2-1$ and $\dim X^* = 1$. In fact, $X^* = D_2^* \subset [D_1]^\vee$.

c): $\bar{m} = 2$.

$$A_2 = \begin{pmatrix} M_2^1 & 0 \\ & \\ \lambda_1 M_2^2 & M_3^2 \end{pmatrix}$$

$X^*|D_1$ is given by

$$\begin{pmatrix} M_2^1 & 0 \\ & \\ 0 & M_3^2 \end{pmatrix}$$

which is a (7×5) matrix, and hence $X^*|D_1 = (D_1^*, D_2^*)$ is a ruled surface. Now $X^*|D_2$ is given by

$$\begin{pmatrix} 0 & M_3^1 \\ & \\ M_2^2 & 0 \end{pmatrix}$$

so $X^*|D_2 = (D_2)_1^\vee \cap [D_1]^\vee$ is equal to the tangent developable $S^1(D_2^*)$ of D_2^* in $[D_1]^\vee$ ([P1], §5).

Hence X^* is determined by the ruled surface (D_1^*, D_2^*) and the developable surface $S^1(D_2^*)$, which also contains D_2^*.

In other words, X^* is determined by $D_1^* \subset [D_2]^\vee$ and $S^1(D_2^*) \subset [D_1]^\vee$. We shall write, with abuse of notation, $X^* = (D_1^*, S^1(D_2^*))$. Hence X^* is a rational (non-normal) 3-dimensional scroll of type (2,2,2) (in the obvious sense) because $S^1(D_2^*)$ is of type (2,2).

Moreover, $X^{**} = (X^*)^\vee_2 = X$; this can be seen by the fact that $X^{**}|D_1^* = D_2$ and $X^{**}|S^1(D_2^*) = D_1$.

The preceding examples are typical for surfaces, in the following sense: If $X = (D_1,D_2)$, then for $m \leqslant \bar{m}$,

$$\dim X_m^\vee = \begin{cases} N+1-2m & m \leqslant d_1 \\ d_2-m & d_1 < m < d_2 \end{cases}$$

For the strict dual $X^* = X_{\bar{m}}^\vee$, there are three possible cases:

a) $d_1 = d_2 = d$: $\bar{m} = d$, $X^* = (D_1^*,D_2^*)$.

b) $d_1 = d$, $d_2 = d+1$: $\bar{m} = d$, $X^* = (D_1^*,S^1(D_2^*))$ is a 3-dimensional (non-normal) scroll of type (d,d,d).

c) $d_1+2 \leqslant d_2$: $\bar{m} = d_2-1$, $X^* = D_2^*$. Moreover, $X_m^\vee$, for $d_1+1 \leqslant m \leqslant d_2-1$, is equal to $S^{d_2-1-m}(D_2^*) \subset [D_1]^\vee$.

Hence X is self-dual only in case a), and only in the cases a) and b) does biduality $(X = X^{**})$ hold.

Example 3. $r = 3$, $\deg X = 6$, $N = 8$.

There are 7 possible cases for the type (d_1,d_2,d_3) of X:

a) $(0,0,6)$
b) $(0,1,5)$
c) $(0,2,4)$
d) $(1,1,4)$

$\left.\right\}$ $\bar{m} = d_3-1$, $X^* = D_3^* \subset [D_1,D_2]^\vee$

e) $(1,2,3)$
f) $(0,3,3)$

$\left.\right\}$ $\bar{m} = d_2$, $X^* = (D_2,D_3)^* \subset [D_1]^\vee$

g) $(2,2,2)$ $\bar{m} = 2$, $X^* = (D_1^*,D_2^*,D_3^*)$.

In the first four cases, the matrix A_m is equivalent to (i.e., defines the same variety as) the matrix

$$\begin{pmatrix} M_{d_1}^{d_1} & 0 & 0 \\ 0 & M_{d_2}^{d_2} & 0 \\ 0 & 0 & M_{d_3}^{d_3-1} \end{pmatrix}$$

In the cases e) and f), the matrix $A_{\bar{m}} = A_{d_2}$ is equivalent to

$$\begin{pmatrix} M_{d_1}^{d_1} & 0 & 0 \\ 0 & M_{d_2}^{d_2-1} & 0 \\ 0 & \lambda_2 M_{d_2}^{d_2} & M_{d_3}^{d_2} \end{pmatrix}$$

The conclusions follow, by considerations as in the previous examples.

Consider now the case g). The matrix $A_{\bar{m}} = A_2$ becomes

$$\begin{pmatrix} M_2^1 & 0 & 0 \\ 0 & M_2^1 & 0 \\ \lambda_1 M_2^2 & \lambda_2 M_2^2 & M_2^2 \end{pmatrix}$$

As in Ex.1 c) we consider the restriction $X^*|D_3$. This is the variety defined by setting $\lambda_1 = \lambda_2 = 0$ in A_2. Hence

$$X^*|D_3 = (D_1^*, D_2^*) \subset [D_3]^\vee.$$

In the same way, we obtain

$$X^*|D_1 = (D_2^*, D_3^*) \subset [D_1]^\vee$$
$$X^*|D_2 = (D_1^*, D_3^*) \subset [D_2]^\vee$$

This shows that $X^* = (D_1^*, D_2^*, D_3^*)$, and hence that X is self-dual.

Since the restriction of X^* to a directrix D_i is the ruled surface defined by the strict duals of the other D_j's, it is natural to ask whether the converse is true.

That is, we want to give a natural definition of the restriction $X^*|(D_2, D_3)$, so that it becomes equal to D_1^*.

To do this, we take for $X^*|(D_2, D_3)$ the points in $\check{P}^N$ corresponding to the hyperplanes that are 2-osculating to X at <u>every</u> point of some generators of (D_2, D_3). This means that we should look at the matrix

$$\begin{pmatrix} M_2^1 & 0 & 0 \\ 0 & M_2^2 & 0 \\ 0 & \lambda_2 M_2^2 & M_2^2 \end{pmatrix},$$

which gives $X^*|(D_2, D_3) = D_1^*$.

Example 4. $r = 3$, $\deg X = 7$, $N = 9$.

The following types are possible:

$$\left.\begin{array}{ll} a) & (0,0,7) \\ b) & (0,1,6) \\ c) & (0,2,5) \\ d) & (1,1,5) \\ e) & (1,2,4) \end{array}\right\} \bar{m} = d_3 - 1, \quad X^* = D_3^* \subset [D_1, D_2]^\vee$$

$$\left.\begin{array}{ll} f) & (0,3,4) \\ g) & (1,3,3) \end{array}\right\} \bar{m} = d_2, \quad X^* = (D_2, D_3)^* \subset [D_1]^\vee$$

h) $(2,2,3)$ $\bar{m} = 2$, $X^* = (D_1^*, D_2^*, S^1(D_3^*))$ is of dimension 4.

The 7 first cases are as before; the only case that presents something new is case h). By looking at the matrix A_2 in this case, we obtain the following:

$$X^*|D_1 = (D_2^*, S^1(D_3^*)) \subset [D_1]^\vee$$
$$X^*|D_2 = (D_1^*, S^1(D_3^*)) \subset [D_2]^\vee$$
$$X^*|D_3 = (D_1^*, D_2^*, D_3^*)$$

$$X^*|(D_1, D_2) = S^1(D_3^*) \subset [D_1, D_2]^\vee$$
$$X^*|(D_1, D_3) = (D_2^*, D_3^*) \subset [D_1]^\vee$$
$$X^*|(D_2, D_3) = (D_1^*, D_3^*) \subset [D_2]^\vee .$$

5. Duality results

Let $X = (D_1, \dots, D_r)$ be a variety of type $(d_1, \dots, d_r)$. Given $(i_1, \dots, i_s) \subset (1, \dots, r)$, we denote by

$$X_{i_1, \dots, i_s} = (D_{i_1}, \dots, D_{i_s});$$

this is a variety of type $(i_1, \dots, i_s)$.

Set $(j_1, \dots, j_{r-s}) = (1, \dots, r) - (i_1, \dots, i_s)$.

With these notations, we give the following

DEFINITION: i) The strict dual curve D_j^* of D_j with respect to $D_{i_1}, \dots, D_{i_{r-1}}$, is the curve

$$D_j^* = (D_j)^\vee_{d_j - 1} \cap [X_{i_1, \dots, i_{r-1}}]^\vee .$$

ii) The restriction of X^* to $X_{i_1,\dots,i_s}$ is the variety $X^*|X_{i_1,\dots,i_s}$ in $\check{\mathbb{P}}^N$ consisting of hyperplanes in $\mathbb{P}^N$ that are $\bar{m}$-osculating to X at every point of some generator of $X_{i_1,\dots,i_s}$.

PROPOSITION 1: Let $X = (D_1,\dots,D_r)$ be a variety of type $(d_1,\dots,d_r)$, and denote by $\bar{m}$ the integer such that $X^* = X^\vee_{\bar{m}}$. For each m, $1 \leq m \leq \bar{m}$, let $i = i(m)$ denote the integer such that $0 \leq i \leq r-1$ and $d_i+1 \leq m \leq d_{i+1}$ (for $i=0$ we set $d_0=0$). If $i \leq r-2$, then

$$\dim X^\vee_m = N+1-rm+\sum_{j=1}^{i}(m-1-d_j).$$

If $i = r-1$, then $\dim X^\vee_m = N-rm+\sum_{j=1}^{r-1}(m-1-d_j) = d_r-m$.

Moreover, if $i \geq 1$, we have

$$X^\vee_m = (X_{i+1,\dots,r})^\vee_m \cap [X_{1,\dots,i}]^\vee.$$

REMARK: For $m = 1$, we obtain $\dim X^\vee_1 \leq N+1-r$. Suppose $r \geq 2$. Then $X^\vee = X^\vee_1$ is a hypersurface if and only if $r = 2$ and $i = 0$, i.e. if and only if X is a 2-dimensional, smooth scroll.

PROPOSITION 2: With the notation of Prop. 1, we have the following description of the strict dual X^* of X.

1) $d_{r-1} = d_r$. Let i be the integer such that $0 \leq i < r-1$, $d_i < d_{i+1} = d_{i+2} = \dots = d_{r-1} = d_r = d$.
Then

$$X^* = X^\vee_d = (D^*_{i+1},\dots,D^*_r) \subset [X_{1,\dots,i}]^\vee.$$

2) $d_{r-1}+1 = d_r$. Let i be such that $0 \leq i < r-1$, $d_i < d_{i+1} = \dots = d_{r-1} = d$.
Then

$$X^* = X^\vee_d = (D^*_{i+1},\dots,D^*_{r-1},S^1(D^*_r)) \subset [X_{1,\dots,i}]^\vee.$$

Hence X^* is a (non-normal) $(r+1-i)$-dimensional scroll, of type $(d,\ldots,d)$.

3) $d_{r-1}+2 \leq d_r = d$. Then
$$X^* = X^{\vee}_{d-1} = D^*_r \subset \{X_{1,\ldots,r-1}\}^{\vee}.$$

<u>COROLLARY</u>: A variety $X = (D_1,\ldots,D_r)$, $X \simeq \mathbb{P}(F) \to \mathbb{P}^N$, of type $(d_1,\ldots,d_r)$, is self-dual if and only if $d_1 = \cdots = d_r$, i.e., if and only if F is semi-stable. In this case we also have the dualities
$$X^*|X_{i_1,\ldots,i_s} = (D^*_{j_1},\ldots,D^*_{j_{r-s}}).$$

<u>Proof of Proposition 1</u>: The results will follow from a study of the generic behavior of the map a^m, or, of the matrix A_m.

First of all, we observe that, if $m \geq d_i+1$, for $i \geq 1$, then all m-th order osculating spaces to X necessarily contain each $\{D_j\}$, $1 \leq j \leq i$, hence $\{X_{1,\ldots,i}\}$; $X^{\vee}_m$ is given by (for $\lambda_r \neq 0$)

$$\begin{pmatrix}
M^{d_1}_{d_1} & & & & & & & \\
 & \ddots & & & & 0 & & \\
 & & M^{d_i}_{d_i} & & & & & \\
 & 0 & & M^{m-1}_{d_{i+1}} & & & & \\
 & & & & \ddots & & & \\
 & & & & & M^{m-1}_{d_{r-1}} & & \\
0 & \cdots & 0 & \lambda_{i+1}M^m_{d_{i+1}} & \cdots & \lambda_{r-1}M^m_{d_{r-1}} & M^m_{d_r}
\end{pmatrix}$$

This shows the last assertion of Prop. 1.

The generic rank of a^m (or A_m) is equal to
$$\sum_{j=1}^{i}(d_j+1)+m(r-i)+1 = rm+1-\sum_{j=1}^{i}(m-1-d_j).$$

Let H be a hyperplane which is m-osculating to X at some (smooth) point $x \in X_t$, for some generator X_t. Then H is m-osculating to X at all points of some linear $(r-2)$-dimensional subspace Y_t of X_t. In fact, H is $(m-1)$-osculating to X at all points of X_t, so $X \cap H$ contains X_t $m-1$ times, and the residual intersection is a (possibly reducible) ruled variety Y of dimension $r-1$. Hence $X \cap H$ contains $Y_t = Y \cap H$ m times, i.e., H is m-osculating to X along Y_t.

From this observation it follows that $X_m^\vee$ is determined by the matrix obtained from A_m by fixing $r-2$ of the λ_i's. That is, we fix a ruled surface $R \subset X$. If $H \in X_m^\vee$, then H is m-osculating to X along $Y_t \subset X_t$ (some t), hence at some point of R_t, since $R_t \cap Y_t \neq \emptyset$.

Assume therefore $\lambda_1 = \cdots = \lambda_{r-2} = 0$. Then the variation in the m-th order osculating spaces along a generator is given by the variation of the matrix

$$\begin{pmatrix} M_{d_{r-1}}^{m-1} & 0 \\ \lambda_{r-1} M_{d_{r-1}}^{m} & M_{d_r}^{m} \end{pmatrix} .$$

If $m \leq d_{r-1}$ (i.e., if $i \leq r-2$), then we see that the m-osculating spaces vary with λ_{r-1} (i.e., are non-constant along R_t). Since they also vary with t, we obtain a 2-dimensional family of m-osculating spaces to X. Each such space has dimension equal to

$$rm - \sum_{j=1}^{i} (m-1-d_j),$$

hence the m-dual variety $X_m^\vee$ has dimension

$$\dim X_m^\vee = N - rm + \sum_{j=1}^{i} (m-1-d_j) - 1 + 2 = N + 1 - rm + \sum_{j=1}^{i} (m-1-d_j).$$

Suppose now $i = r-1$, so that $d_{r-1} < m < d_r$. Then the variation of the m-osculating spaces is given by the variation of the matrix $M_{d_r}^m$, hence is independent of the λ_i's (i.e., the m-osculating spaces are constant along the generators of X).

Moreover, these spaces are the ones spanned by the $[D_j]$'s, $1 \leq j \leq r-1$, and the m-osculating spaces to D_r. In other words, in this case

$$X_m^\vee = S^{d_r-m-1}(D_r^*) \subset [X_{1,\ldots,r-1}]^\vee$$

is the (d_r-m-1)-th osculating developable of the strict dual curve $D_r^* \subset [X_{1,\ldots,r-1}]^\vee$. In particular $\dim X_m^\vee = d_r-m$.

Proof of Proposition 2:

Case 1): $d_{r-1} = d_r = d$.

Considering the matrix A_m we observe that $\bar{m} = d$ holds, and that every d-osculating space contains the space $[D_1,\ldots,D_i] = [X_{1,\ldots,i}]$. Moreover, for each k, $i+1 \leq k \leq r$, we have

$$X^*|D_k = (D_{i+1}^*,\ldots,D_{k-1}^*,D_{k+1}^*,\ldots,D_r^*) \subset [X_{1,\ldots,i}]^\vee \cap [D_k]^\vee.$$

Thus we wee that

$$X^* = X_d^\vee = (D_{i+1}^*,\ldots,D_r^*) \subset [X_{1,\ldots,i}]^\vee$$

holds.

Case 2): $d_{r-1}+1 = d_r$.

We see that $\bar{m} = d_{r-1} = d$ holds, and that every d-osculating space contains $[X_{1,\ldots,i}]$. Moreover, for each k, $i+1 \leq k \leq r-1$, we have

$$X^*|D_k = (D_{i+1}^*,\ldots,D_{k-1}^*,D_{k+1}^*,\ldots,D_{r-1}^*,S^1(D_r^*)) \subset [X_{1,\ldots,i}]^\vee \cap [D_k]^\vee,$$

where $S^1(D_r^*) \subset [X_{1,\ldots,r-1}]^\vee$ is the tangent developable of D_r^*.

For $k = r$, we have

$$X^*|D_r = (D_{i+1}^*,\ldots,D_r^*) \subset [X_{1,\ldots,i}]^\vee \cap [D_r]^\vee.$$

It follows that

$$X^* = X_d^\vee = (D_{i+1}^*,\ldots,D_{r-1}^*,S^1(D_r^*)) \subset [X_{1,\ldots,i}]^\vee.$$

To see that X^* is (non-normal) of type $(d,\ldots,d)$, it suffices to show that $S^1(D_r^*)$ is (non-normal) of type (d,d). This follows from the following general lemma.

<u>LEMMA</u>: Let $D \subset \mathbb{P}^d$ be a rational normal curve of degree d $(d \geq 3)$. Then the m-osculating develpable $S^m(D)$ $(m \leq d-2)$ of D is a (non-normal) variety of type $(d-m,\ldots,d-m)$ (in $\mathbb{P}^d$). In other words, the bundle $P_D^m(1)$ is isomorphic to $\bigoplus_{i=0}^{m} \mathcal{O}_{\mathbb{P}^1}(d-m)$, i.e., $P_D^m(1)$ is semistable.

<u>Proof</u>: It suffices to exhibit $m+1$ independent sections of degree $d-m$. If D is given by

$$x_i = u^i v^{d-i} , \qquad i = 0,1,\ldots,d,$$

then

$$s_k = \frac{\partial^m x}{\partial u^{m-k} \partial v^k} , \qquad k = 0,\ldots,m,$$

are the desired sections.

<u>Case 3)</u>: This has already been shown in the proof of Prop. 1, by observing $\bar{m} = d_r - 1$.

<u>REMARK</u>: Suppose $d_1 > 0$ (i.e. X is a rational normal scroll). Then biduality $(X^{**}=X)$ holds if and only if $d_1 = \cdots = d_{r-1}$ and $d_r = d_{r-1}$ or $d_r = d_{r-1}+1$, in other words, if and only if X is a <u>general</u> scroll (in the sense of Ghione [G]). In the balanced case $(d_1 = \cdots = d_r)$ we have already seen that X is in fact self-dual. In the second case we have

$$X^{**} | D_j^* = X_{i_1,\ldots,i_{r-1}} \qquad (j \neq r)$$

and

$$X^{**} | S^1(D_r^*) = X_{1,\ldots,r-1} ,$$

from which we conclude: $X^{**} = X$.

In all other cases, there is an i, $1 \leq i \leq r-1$, such that $d_i + 1 \leq \bar{m} \leq d_{i+1}$; hence

$$X^* \subset [X_{1,\ldots,i}]^{\vee}.$$

Consequently, X^{**} is the <u>cone</u> over X with vertex $[X_{1,\ldots,i}]$.

6. Curves in Grassmann varieties

A ruled variety can also be considered as a variety in a Grassmann variety. If $X \subset \mathbb{P}^N$ is of type $(d_1, \ldots, d_r)$, defined by $V_{\mathbb{P}^1} \to F = \bigoplus_{i=1}^{r} O_{\mathbb{P}^1}(d_i)$, then that also defines a morphism

$$\mathbb{P}^1 \to G = \mathrm{Grass}_r(V).$$

Let $\Gamma \subset G$ denote the image of this map.

The m-dual variety $\Gamma_m^\vee \subset G^\vee = \mathrm{Grass}_r(V^\vee)$ is defined as (the closure of) the set of $(N-r)$-spaces that contain an m-osculating space to Γ [P2, §2]. These latter spaces are defined by the image of

$$b^m : V_{\mathbb{P}^1} \to P^m_{\mathbb{P}^1}(F) = \oplus_{i=1}^{r} P^m_{\mathbb{P}^1}(d_i).$$

The image of b^m is defined by the matrix

$$B_m = \begin{pmatrix} M^m_{d_1} & & 0 \\ & \ddots & \\ 0 & & M^m_{d_r} \end{pmatrix}$$

of rank $\beta_m = r(m+1) - \sum_{j=1}^{i} (m-d_j)$, where $i = i(m)$ is the integer such that $0 \leq i \leq r-1$ and $d_i \leq m < d_{i+1}$ (again we set $d_0=0$). The m-dual variety $\Gamma_m^\vee$ is defined whenever $\beta_m \leq N+1-r$.

The strict dual variety of Γ is $\Gamma^* = \Gamma^\vee_{\overline{m}} \subset G^\vee = \mathrm{Grass}_r(V^\vee)$, where $\overline{m}$ is the smallest integer m such that β_m is maximal subject to $\beta_m \leq N+1-r$.

If $d_1 > 0$ and $m \leq d_1 - 1$, then $\beta_m = r(m+1)$ and $\Gamma_m^\vee$ is defined. In this case, $\mathrm{Ker}(b^m)^\vee$ is the sum of the bundles of principal parts of order d_i-1-m of the "dual" of $O_{\mathbb{P}^1}(d_i)$ (interpreted in terms of the curve D_i and its strict dual D_i^*, using duality for the corresponding short exact sequences ([P1], 5.2)).

PROPOSITION 3: Let $\Gamma \subset G$ correspond to $X \subset \mathbb{P}^N$ of type $(d_1,\dots,d_r)$, $r \geq 2$.

1) If $d_1 = \dots = d_r = d$, then $\Gamma^* = \Gamma^\vee_{d-1}$, $\dim \Gamma^* = 1$, and $X^* = X^\vee_d$ is the total space of Γ^*.

2) If $d_1 = \dots = d_{r-1} = d$, $d_r = d+1$, then $\Gamma^* = \Gamma^\vee_{d-1}$, $\dim \Gamma^* = 2$, and $X^* = X^\vee_d$ is the total space of Γ^*.

3) If $d_1 < d_2 = \dots = d_{r-1} = d$, $d_r = d+1$, then $\Gamma^* = \Gamma^\vee_{d-1}$, $\dim \Gamma^* = 1$.

In all other cases, $\Gamma^* = \Gamma^\vee_{\bar{m}}$, where $\bar{m} < d_r-2$ if $r > 2$; if $r = 2$, $d_{r-1} \leq d_r-2$, then $\bar{m} = d_r-2$, and Γ^* is the curve corresponding to the tangent developable of D_2^*.

Proof: Assume $d_r = d$, and that $\Gamma^\vee_{d-1}$ is defined. Then (with i as before),

$$\beta_{d-1} = rd - \sum_{j=1}^{i} (d-1-d_j) \leq N+1-r = \sum_{j=1}^{r} d_j ,$$

so

$$rd-i(d-1) \leq \sum_{j=i+1}^{i} d_j \leq (r-i)d.$$

Hence $i = 0$, and $d-1 < d_1 \leq d_2 \leq \dots \leq d$. Therefore $d_1 = \dots = d_r = d$. In this case, clearly $\bar{m} = d-1$, and Γ^* is the curve whose total space is $X^* = (D_1^*,\dots,D_r^*)$.

Assume $d_{r-1} = d$, $d_r = d+1$, and that $\Gamma^\vee_{d-1}$ is defined (clearly $\bar{m} \leq d-1$). Then

$$\beta_{d-1} = rd - \sum_{j=1}^{i} (d-1-d_j) \leq \sum_{j=1}^{r} d_j ,$$

so that we get

$$rd-i(d-1) \leq \sum_{j=i+1}^{r} d_j \leq (r-i)d+1.$$

Hence $i \leq 1$. Suppose $i = 0$. Then $d-1 < d_1$, so $d_1 = \cdots = d_{r-1} = d$, $d_r = d+1$, $\beta_{d-1} = rd$, and $\mathrm{Ker}(b^{d-1})$ has rank

$$N+1-rd = (r-1)d+d+1+r-rd = r+1.$$

Hence $\Gamma^* = \Gamma^{\vee}_{d-1}$ is ruled by lines, so $\dim \Gamma^* = 2$.
The matrix B_{d-1} in this case is just the one associated with $X^* = X^{\vee}_{d} = (D^*_1,\ldots,D^*_r,S^1(D^*_r))$.

If $i = 1$ (hence $r \geq 3$), then $\beta_{d-1} = (r-1)d+d_1+1$ and the rank of $\mathrm{Ker}(b^{d-1})$ is $N+1-\beta_{d-1} = r$. Hence $\Gamma^* = \Gamma^{\vee}_{d-1}$ is a curve.

Suppose finally that $d_{r-1} \leq d_r-2$, and that $\Gamma^{\vee}_m$ is defined for some $m \geq d_r-2$. Then $i = r-1$, and we must have

$$r(m+1)-(r-1) \leq d_r \;, \quad r+m \leq d_r \leq m+2.$$

Hence $r \leq 2$, so $r = 2$, and $m = d_2-2$. Then clearly Γ^* is the curve in $G^{\vee}$ corresponding to the tangent developable $S^1(D^*_2)$ of D^*_2 in $[D^*_2]$.

<u>REMARK</u>: Set $F^*_m = (\mathrm{Ker}\ b^m)^{\vee}$. From what we have seen, F^*_m is the direct sum of bundles of principal parts on the curves $D^*_1,\ldots,D^*_r$. We can write

$$F^*_m = \bigoplus_{j=1}^{r} P_{\mathbb{P}^1}^{d_j-m-1}(d_j)$$

(if $m+1 > d_j$, we set the bundle equal to 0).
From a previous lemma, we know that $P_{\mathbb{P}^1}^{d_j-m-1}(d_j)$ is balanced: it splits into d_j-m copies of $O_{\mathbb{P}^1}(d_j-(d_j-m-1)) = O_{\mathbb{P}^1}(m+1)$. Hence, for m such that $m \leq d_1-1$, the $(\sum_{j=1}^{r} d_j-rm)$-bundle F^*_m is balanced of type $(m+1,\ldots,m+1)$. Hence $\Gamma^{\vee}_m$ is defined by a balanced bundle F^*_m (recall that $\Gamma^{\vee}_m$ is the image of $\mathrm{Grass}_r(F^*_m) \subset \mathbb{P}^1 \times \mathrm{Grass}_r(V^{\vee})$ in $G^{\vee} = \mathrm{Grass}_r(V^{\vee})$), even though the bundle F defining Γ was not necessarily balanced. Moreover, the case F balanced corresponds to F^*_{d-1} balanced and rank $F^*_{d-1} = r$; this is precisely when X and Γ are self-dual.

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